Energy spectra of strongly stratified and rotating turbulence

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Turbulence under strong stratification and rotation is usually characterized as quasi-two-dimensional turbulence. We develop a ''quasi-two-dimensional'' energy spectrum which changes smoothly between the Kolmogorov $-\frac{5}{3}$ law (no stratification), the -2 scalings of Zhou [Phys. Fluids **7**, 2092 (1995)] for the case of strong rotation, as well as the -2 scalings for the case of strong rotation and stratification. For strongly stratified turbulence, the model may give the -2 scaling predicted by Herring [Meteorol. Atmos. Phys. **38**, 106 1988], and the $-\frac{5}{3}$ scaling indicated by some mesoscale observations. [S1063-651X(98)00705-3]

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The important applications of turbulence under strong stratification and rotation in geophysics and engineering are well documented (Refs. [1–5]). As an example, large-scale flows in oceans and the earth's atmosphere are known to be almost two dimensional, but not exactly so. Typically, one characterizes these types of flows as quasi-two-dimensional turbulence. In this note, we discuss the development of the spectra for turbulence subject to strong (stable) stratification and rotation.

We define the asymptotic regimes of geophysical dynamics in terms of the following nondimensional parameters. Let H be the vertical (spectral) length scale, L the horizontal length scale, and U_h a characteristic horizontal velocity scale. Then the spectral aspect ratio can be defined as $a=H/L$. We define the Froude number based on horizontal and vertical scales:

$$F_h = U_h/LN_0 \equiv 1/N, \quad F_v = U_h/HN_0 = F_h/a. \tag{1}$$

The classical Rossby and anisotropic Rossby numbers are defined as follows:

$$Ro = U_h/Lf_0 \equiv 1/f, \quad Ro_a = Ro \times a. \tag{2}$$

Here N_0 is the Brunt-Väisälä frequency for the constant stratification gradient and $f_0 = 2\Omega_0$ is the Coriolis parameter (Ω_0 is the frequency of background rotation); the vertical axis is chosen to be aligned with the axis of rotation and the mean stratification gradient. The governing flow equations are three-dimensional (3D) Euler-Boussinesq equations for rotating stratified fluids with zero-flux boundary conditions in the vertical direction. Such boundary conditions imply zero tangential stress on the vertical boundaries.

The Burger number characterizes the relative importance of the effects of rotation and stratification [10]:

$$Bu = Ro_a^2/F_h^2 \equiv Ro^2/F_v^2 \equiv N_0^2 a^2/f_0^2, \tag{3}$$

with $Bu \ll 1$ corresponding to rotation-dominated flows and $Bu \gg 1$ corresponding to stratification-dominated flows. Also, a measure of the relative importance of (stable) stratification versus rotation is the internal radius of deformation Λ. The internal (Rossby) radius of deformation Λ is defined as

$$\Lambda = N_0 H/f_0, \tag{4}$$

so that $Bu = (\Lambda/L)^2$. When $L \gg \Lambda$, the flow is organized in quasi-2D vertical columns and when $L \ll \Lambda$, the flow is organized in thin horizontal layers with a strong vertical variability.

From the mathematical analysis (Babin and co-workers, [6–9]), we constructed Fig. 1 to illustrate the global picture of geophysical dynamics at small Froude and/or small Rossby regimes. Since we are not taking $a \to 0$, either F_h or F_v can be used in describing asymptotic regimes. Then Fr denotes either of these numbers.

When only strong rotation exists (Fig. 1, vertical axis), any solution of the initial value problem for the 3D Euler-Boussinesq–Navier-Stokes system can be split into two parts. The first component is a solution of the two-dimensional (2D) barotropic Euler-Boussinesq–Navier-Stokes system with vertically averaged initial data. The dy-

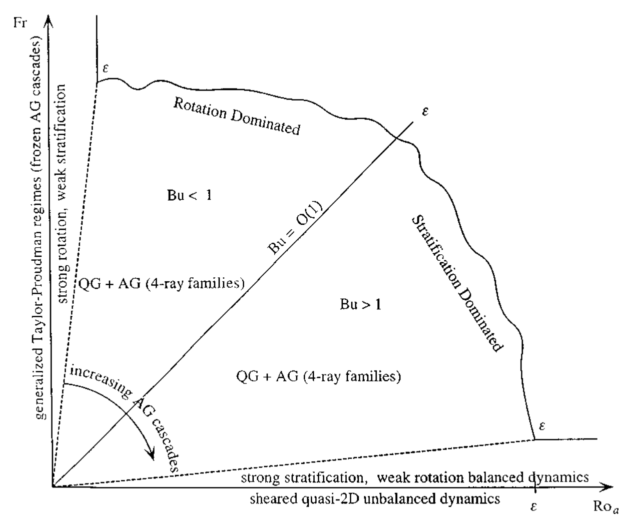

FIG. 1. Geophysical dynamics: the global picture for small Froude or small Rossby regimes.

namics of the second part that describes vertical variability is called ageostrophic in this limiting context. In this asymptotic regime, it is exactly solved in terms of 2D dynamics of vertically averaged fields. The error of the splitting is of the order of the anisotropic Rossby number [defined by Eq. (2)], a very small number in many situations (Babin and co-workers [6,8,9]). Energy cascades for the ageostrophic modes are completely frozen in the vertical direction and the ageostrophic dynamics is pure *phase turbulence*. In pure phase turbulence, the amplitudes of the ageostrophic modes remain constant in absolute values; turbulent dynamics are restricted to the phases of the ageostrophic modes. The ageostrophic field is phase locked to phases associated with vertically averaged vertical vorticity and vertical velocity, which are advected by 2D turbulence of vertically averaged fields. There is no slaving of the amplitudes of ageostrophic modes by the 2D turbulence, only phase locking.

When stratification is present, the cascades of ageostrophic modes (AG) become ''unfrozen.'' As stratification increases, the direct cascade of ageostrophic energy from large scales to small scales increases (Babin *et al.*, [8]). When both rotation and stratification effects are of the same order of magnitude, the situation called Burger one regime [10], Babin *et al.* [7–8] established the splitting between 3D quasigeostrophic (QG) and reduced ageostrophic fields using the Craya-Herring cyclic bases [11]. In these bases the ageostrophic modes are characterized in terms of the divergent velocity potential (horizontal divergence) and the geostrophic departure and/or thermal wind unbalance (e.g., [8]). The QG modes inversely transfer the vortical (rotational) energy upscales. On the other hand, direct energy cascades of the AG field provide a mechanism for *nonlinear geostrophic adjustment*. This is fundamentally different from the rotation-dominated regimes where AG cascades are frozen. The nonlinear geostrophic adjustment mechanism is indeed the capacity of the AG dynamics to transfer energy to smaller scales and eventually dissipate its inertio-gravitational energy [12]. Direct cascades of energy of the ageostrophic modes indicate that the observed $-\frac{5}{3}$ power law is the spectrum of internal gravity waves with direct energy cascade to large wave numbers (small scales).

In order to infer the form of the inertial-range spectrum $E(k)$ for different asymptotic regimes, it is necessary to estimate the magnitude for the triple velocity correlations. In general, τ_3, the time scale for the decay of the triple correlations responsible for inducing turbulent spectral transfer, may depend on any relevant turbulence parameters [13,14]. When energy is conserved by the nonlinear interaction and a local cascade has been assumed, the energy flux, which equals the dissipation rate ϵ, is independent of wave number k. Local cascade also implies that ϵ is explicitly proportional to τ_3, and depends on the wave number and on the power of the omni-directional energy spectrum. A simple dimensional analysis leads to

$$\epsilon = A^2 \tau_3(k) k^4 E^2(k), \tag{5}$$

where A is a constant. When the time scale for triple decorrelation is simply given by the nonlinear time $\tau_3(k) = \tau_{nl} = [k^{3/2} E^{1/2}(k)]^{-1}$, the classical Kolmogorov spectrum is recovered.

At asymptotic limits of strong rotation, strong stratification, and the limit of strong rotation and stratification, there are two disparate time scales. The difference in time scales and anisotropies in length scales are crucial for the mathematical analysis of Babin *et al.* [6–9], and is the basic requirement for the methodology of our phenomenological analysis [5,13–15]. The major difficulty encountered in understanding the dynamics of geophysical flows is the influence of the oscillations (inertio-gravity waves) generated by the rotation and stratification. This effect leads to the modification of the spectral time for energy transfer down scales.

We recall that the dispersion relation for inertio-gravity waves is given by the formula $\omega_k^2 = N_0^2 (k_h^2/k^2) + f_0^2 (k_3^2/k^2)$, where $k = (k_1, k_2, k_3)$ is the wave vector, and $k_h^2 = k_1^2 + k_2^2$ and $k^2 = k_1^2 + k_2^2 + k_3^2$ (axes of rotation and gravity are along the vertical axis $e_3 = [0,0,1]$). As above, we define the vertical and the horizontal spectral scales as $H = 1/|k_3|$, $L = 1/k_h$. If $a = H/L = k_h/|k_3|$ is the ratio of these length scales, then $\omega_k^2 = (f_0/\alpha)^2 + (N_0/\beta)^2$, where $\beta = \sqrt{a^{-2}+1}$ and $\alpha = \sqrt{a^2+1}$. Then, the spectral Burger number [Eq. (3)] is a ratio of the spectral stratification frequency N_0/β and the spectral rotation frequency f_0/α: $Bu = (N_0/\beta)^2/(f_0/\alpha)^2 = (N_0\alpha)^2/(f_0\beta)^2 = N_0^2 H^2 / f_0^2 L^2$.

We first give a brief treatment for the strongly rotating turbulence case. In a regime of high Reynolds numbers and low Rossby numbers, turbulence is characterized by a short time scale $\tau_f = \alpha/f_0$, where $\alpha = \sqrt{a^2+1}$ and $f_0 = 2\Omega_0$. A direct application of $\tau_3 = \tau_f$ results in the energy spectrum for turbulence subject to strong rotation:

$$E(k) = C_f (\epsilon f_0/\alpha)^{1/2} k^{-2}, \tag{6}$$

where C_f is a constant [14]. The introduction of the aspect ratio into the time scale is an improvement over our previous phenomenological analysis [14,15] since the model can now distinguish the anisotropic nature of rotating flow.

For turbulence in the Burger one regime, the same procedure, namely setting $\tau_3 = \tau_{fN}$, leads to the energy spectrum for turbulence subject to strong rotation and stratification:

$$E(k) = C_{fN} [\epsilon \sqrt{(N_0/\beta)^2 + (f_0/\alpha)^2}]^{1/2} k^{-2}, \tag{7}$$

where $\beta = \sqrt{a^{-2}+1}$.

Based on the analogy between the rotating and stratified turbulence [2], previous results can be extended to the cases of strongly stratified turbulence. Substituting $\tau_3 = \tau_N$ leads to

$$E(k) \sim (\epsilon N_0/\beta)^{1/2} k^{-2}. \tag{8}$$

This reduces to a result found previously by Herring [16] and Kimura and Herring [17] for the isotropic case.

We note, however, that based on experimental findings by Dickey and Mellor [18], some further extensions may be needed in the cases of strongly stratified turbulence, and probably also for the case of stratified and rotating turbulence. We shall restrict our discussion below to strongly stratified turbulence for brevity. The results can be extended trivially to include the rotation effect by using τ_{fN} instead of τ_N. For a strongly stratified flow, the energy-transfer process may be modified in two ways. First, the effect of the internal waves is reflected in the reduced time scales for the triple correlations. This effect reduces the rate of the direct energy transfer down scales and leads to the spectrum predicted by Herring [16]. Second, the effect of the internal waves may lead to a direct reduction in the energy flux, and this reduction is called the ''energy radiation rate.'' Indeed, for moving-grid-generated turbulence, Dickey and Mellor [18] showed a clear break in the decay rate of the turbulence energy when the buoyancy effects become active. This break indicates the collapse of three-dimensional turbulence at nearly all scales. The interpretation given by Dickey and Mellor [18] is that the nonlinear energy transfer now has the general form

$$\epsilon=(u^3/l)-CN_0^3l^2. \tag{9}$$

Here u is the rms of turbulent velocity, l is the integral scale obtained by integrating the longitudinal velocity autocorrelation, and C is a constant estimated experimentally as 1.910^{-2} [18].

Including the ''energy radiation term'' of Dickey and Mellor only requires a very minor modification to our procedure. Note that Eq. (5) can be rewritten as

$$\epsilon'=A^2\tau_3(k)k^4E^2(k), \tag{10}$$

where we have introduced the effective dissipation rate $\epsilon'=\epsilon+CN_0^3l^2$. We remark that when the first effect of stratification is weak ($\tau_3\sim\tau_{nl}$), the energy spectrum modified by the second effect of stratification takes the form

$$E(k)\sim(\epsilon')^{2/3}k^{-5/3}. \tag{11}$$

In general, the lifetime of triple correlations in rotating and stratified turbulence might be more accurately treated by taking into account the possibility that these correlations decay because of the influence of both wave propagation and nonlinear triadic interactions [14]. The simple choice

$$\frac{1}{\tau_3(k)}=\frac{1}{\tau_{nl}(k)}+\frac{1}{\tau_E(k)} \tag{12}$$

satisfies the appropriate limiting cases: $\tau_3(k)\rightarrow\tau_{nl}$ without external agencies, and $\tau_3(k)\rightarrow\tau_E$ with external agencies. Here τ_E may be taken as τ_f, τ_N, or τ_{fN}.

We now find that the general energy spectrum for strongly stratified turbulence takes the form

$$E(k)=Z'^2A^{-4/3}\epsilon'^{2/3}k^{-5/3}, \tag{13}$$

where Z' is given by

$$Z'=\tfrac{1}{2}\left(\sqrt{Y'}+\sqrt{-Y'+2\sqrt{Y'^2+4Z_0'}}\right), \tag{14}$$

where

$$Y'=\sqrt[3]{\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+\left(\frac{4Z_0'}{3}\right)^3}}+\sqrt[3]{\tfrac{1}{2}-\sqrt{\tfrac{1}{4}+\left(\frac{4Z_0'}{3}\right)^3}}. \tag{15}$$

The parameters are $A=C_K^{-3/4}$ and $Z_0'=[Ak_E'/k]^{2/3}$. Again, depending on the situation, k_E' may take values from $k_N'=(N_0^3/\epsilon')^{1/2}$ (for strongly stratified flows), and $k_{fN}'=\{[(N_0/\beta)^2+(f_0/\alpha)^2]^{3/2}/\epsilon'\}^{1/2}$ (for stratified and/or rotating flows). These equations reduce to the classical Kolmogorov ''$-\frac{5}{3}$'' spectrum when $Z_0'\rightarrow 0$ (so that $Z'\rightarrow 1$), and to our strongly stratified or rotation and/or stratification modified ''-2'' spectrum when $Z_0'\rightarrow\infty$ (so that $Z'\rightarrow Z_0'^{\,1/4}$). Alternatively, the scaling of the energy spectrum in the strongly stratified case may remain as $-\frac{5}{3}$ (as pointed out already by $\tau_3\sim\tau_{nl}$). For intermediate strength of the stratification (or rotation and/or stratification) the spectrum varies smoothly between these two limiting forms, according to the increase of the controlling parameter Z_0' with increasing ratio k_E'/k.

We note the difference between the energy spectra in a 2D case [19] and a quasi-2D case. Based on statistical turbulence theory, Herring [16] found that it is quite difficult to justify 2D spectra of $-\frac{5}{3}$ (for scales larger than the energy injection scale) and -3 spectra (for scales smaller than the energy injection scale) for a stratified flow. The reason is that the triple-moment relaxation is dominated by waves. The fact that the time scale for triple velocity correlation is dominated by faster time scale forms the foundation for our development and the work of Herring [16].

We conclude this paper by noting that our energy spectrum, in its most general form, would lead to an energy spectrum that changes smoothly between the Kolmogorov $-\frac{5}{3}$ law (no stratification), the -2 scalings of Zhou [14] for the case of strong rotation, as well as the -2 scalings for the case of both strong rotation and stratification. For strongly stratified turbulence, the model may give the -2 scaling predicted by Herring [16] and the $-\frac{5}{3}$ scaling indicated by some observations at mesoscales.

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